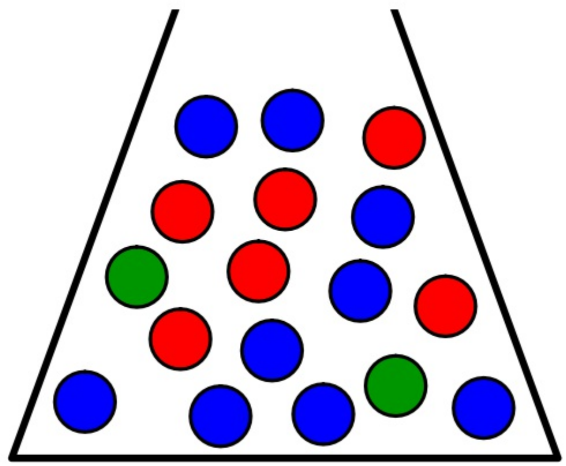
Same or Different?

Exploration
(Unit 1/2 Math Methods)



Introduction to the (1,2) game.

A	B
SAME	DIFF
7, 5, 6	22, 16, 23
10, 12, 8	22, 24, 8
48	115

163

Empirical data

$$\boxed{P_r(\varepsilon) = \frac{n(\varepsilon)}{n(S)}}$$

$\varepsilon \equiv \text{SAME}$

$$P_r(\text{SAME}) = \frac{n(\text{SAME})}{n(\text{TOTAL})}$$

$$= \frac{48}{163}$$

$$= 0.294$$

or 29.4%

$$Pr(\text{DIFF}) = 1 - Pr(\text{SAME})$$

(COMPLEMENT)

$$= 1 - 0.294$$

$$= 0.706$$

(2, 2) game

SAME	DIFF
11, 13, 15	19, 21, 20
15, 15,	25, 33,

Setting the challenge. Find fair games empirically.

GAME	Pr(S)	Pr(D)
(2, 1)	0.294	0.706
(2, 2)	0.369	0.631

CHALLENGE:

SEARCH FOR A FAIR GAME.

Combinations found empirically that we could

a) support with the computer software,

b) confirm using probability theory

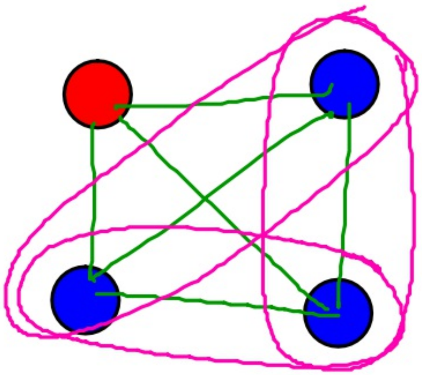
COMBINATIONS WE WANT TO TEST FOR FAIRNESS.

$(2,3)$ $(3,3)$ $(2,4)$ $(6,4)$

$(3,4)$ $(3,5)$ $\boxed{(3,6)}$ $(10,5)$

$(4,8)$ $\boxed{(1,3)}$ $\boxed{\cancel{(12,8)}}$ $\boxed{\cancel{(9,13)}}$

Using probability theory to prove fairness



6 possible pairs.

3 pairs are same.

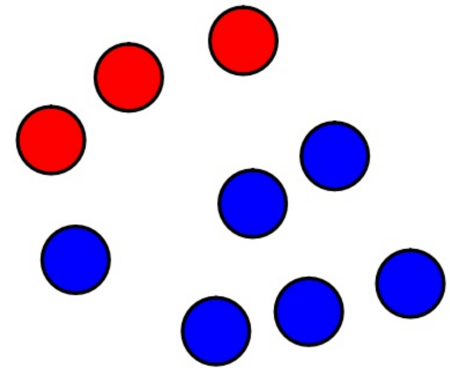
$$Pr(\text{same}) = \frac{n(\text{same})}{n(\text{total})} = \frac{3}{6} = \frac{1}{2}$$

Same $\quad {}^3C_2$

total $\quad {}^4C_2$

$$Pr(\text{same}) = \frac{{}^3C_2}{{}^4C_2} = \frac{3}{6} = \frac{1}{2}$$

$(3, 6)$?



same ${}^6C_2 + {}^3C_2$

total ${}^9C_2$

$$Pr(\text{same}) = \frac{{}^3C_2 + {}^6C_2}{{}^9C_2}$$

$$= \frac{1}{2}$$

Games that appeared fair on the computer were proved to be close, *but not perfect*.

$(8, 12)$

$$\Pr(\text{same}) = \frac{{}^{8}C_2 + {}^{12}C_2}{{}^{20}C_2}$$

$$= \frac{47}{95}$$

close!

$(9, 13)$

$$\Pr(\text{same}) = \frac{38}{77}$$ close!

$$(6, 9) \rightarrow \frac{17}{35}$$ close!

These combinations make perfectly fair games:

$(1, 3) \quad (3, 6) \quad (6, 10) \quad (10, 15)$

Pairs of triangular numbers provide perfectly fair games.

$$1, 3, 6, 10, 15, 21, \boxed{28, 36} \ldots\ldots$$

$$\Pr(\text{same}) = \frac{{}^{28}C_2 + {}^{36}C_2}{{}^{64}C_2} = \frac{1}{2}$$

Can we add a 3rd colour?

n_1 blue

n_2 red

n_3 green

$$\Pr(\text{same}) = \frac{{}^{n_1}C_2 + {}^{n_2}C_2 + {}^{n_3}C_2}{{}^{(n_1+n_2+n_3)}C_2}$$

A new challenge:

Find (n_1, n_2, n_3) such that $\Pr(\text{same}) = \frac{1}{2}$

Using Excel to search for fair 3-colour games

	Number of Blocks	Pairs
colour 1	2	1
colour 2	3	3
colour 3	11	55
colour 4	0	0
colour 5	0	0
colour 6	0	0
Total	16	

Numerator
59
120
Denominator

Pr(Same)= 0.49167

	A	
3	n	$^nC_2$

= Combin (A3, 2)

$\boxed{Pr}$

Each "fair pair" is the beginning of a fair trio:

There is a pattern for the 3rd number

$(1, 3, ?)$ $\qquad 3 \times 4 - 3 = 9$

$(\underline{3, 6}, 19)$ $\qquad 6 \times 4 - 5$

$(\underline{6, 10}, 33)$ $\qquad 10 \times 4 - 7$

$(\underline{10, 15}, 51)$ $\qquad 15 \times 4 - 9$

Here is another pattern for the 3rd number:

$(1, 3, ?)$ $\quad (1+3)\times 2+1$

$(\underline{3, 6}, 19)$ $\quad (3+6)\times 2+1$

$(\underline{6, 10}, 33)$ $\quad (6+10)\times 2+1$

$(\underline{10, 15}, 51)$ $\quad (10+15)\times 2+1$

Four colours? Yes.

4th number

$(1, 3, 9, 27)$ $\quad 9 \times 3 = 27$

$(3, 6, 19, 57)$ $\quad 19 \times 3 = 57$

$(6, 10, 33, 99)$ $\quad 33 \times 3 = 99$

Or the same pattern as before.

Add all, then double, then add 1.

$$(1, 3, 9, 27) \qquad (1+3+9)\times 2+1 = 27$$

$$(3, 6, 19, 57) \qquad (3+6+19)\times 2+1 = 57$$

$$(6, 10, 33, 99) \qquad (6+10+33)\times 2+1 = 99$$

Five colours?

$$(1, 3, 9, 27, ?)$$

$$(1+3+9+27)\times 2+1 = 81$$

$$\therefore (1, 3, 9, 27, 81) \Rightarrow \frac{1}{2}$$

There is a unique pattern here.

	Number of Blocks	Pairs
colour 1	3	3
colour 2	6	15
colour 3	19	171
colour 4	57	1596
colour 5	171	14535
colour 6	0	0
Total	256	

Numerator
16320
32640
Denominator

Pr(Same)= 0.5

CONJECTURE:

A fair game exists for any number of colours, n, where the combination of blocks is

$$\left(3^0, 3^1, 3^2, 3^3, 3^4, \ldots, 3^{n-1}\right)$$

Demonstration: Using the patterns we have found to produce solutions for the 6-colour game.

N = 1

Choose any term in the sequence of triangular numbers.

	Number of Blocks	Pairs
colour 1	1	0
colour 2	3	3
colour 3	9	36
colour 4	27	351
colour 5	81	3240
colour 6	243	29403
Total	364	

Numerator
33033
66066
Denominator

Pr(Same)= 0.5

N = 8

The 8th triangular number is 36.

	Number of Blocks	Pairs
colour 1	36	630
colour 2	45	990
colour 3	163	13203
colour 4	489	119316
colour 5	1467	1075311
colour 6	4401	9682200
Total	6601	

Numerator
10891650
21783300
Denominator

Pr(Same)= 0.5

N = 20

The 20th triangular number is 210.

	Number of Blocks	Pairs
colour 1	210	21945
colour 2	231	26565
colour 3	883	389403
colour 4	2649	3507276
colour 5	7947	31573431
colour 6	23841	284184720
Total	35761	

Numerator
319703340
639406680
Denominator

Pr(Same)= 0.5