Slide 1

How Can Solving the World's Hardest Problem Inform Mathematics Teaching?

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Introduction

Thank you for choosing to come to this keynote address. Given the natural beauty of this place, I believe you had at least one other choice.

For my part I recognise the extra responsibility associated with being invited to deliver a keynote and for some time now I have been pondering what credentials I have that would have brought this responsibility on me. I have come to the conclusion that it must be...

... age.

My age was brought home to me not long ago at the local shopping centre. I was dragged along to push the trolley for my Princess. Not such a bad thing, but it was supposed to be a shortish excursion away from the computer, so I certainly didn't dress for the occasion. I was comfortable; not the best jeans and my very favourite, very bulky, zipper up the front cardigan. Our supermarket has an exit through a side corridor to a back car park. Shopping done we headed in that direction, but hadn't gone ten steps when:

Oh I forgot...

I volunteered to wait in the side corridor and meandered into it slothfully pushing a relatively co-operative cart. I backed into a wall, slouched on the trolley and settled down to let my mind wander back to what I had left on the computer.

Suddenly around the corner from the car park direction came two young men in Armaguard uniform. One was pushing a cart of some sort and was very focussed on where they were going. The other was hovering around him, hand on the Smith & Weston holstered on his hip, eyes darting in all directions. He spied me ... and I suspect sized me up in a millisecond as non-threatening. As they flashed past he greeted me with:

G'day digger!

G'day digger!

I was flabbergasted. Couldn't he see it was my father who fought in the war - and even that was the second one!

However, it wasn't actually the event itself that made me realise I had earned my age credentials. It was actually my reaction to it. Before my Princess turned up with the missing item, I realised that in my mind my first response had actually been:

Why you young whipper-snapper!

Story Telling

So, credentials established, one thing old people are allowed to do is tell stories, and that, happens to fit very well with the professional role I have fulfilled since around 1993.

Slides 2 - 4

Today, some of the stories will be from classrooms and some I will take from this book, which is the...

Fermat Book

...second greatest story ever told.

Slide 5

Let me take a moment now to establish the credentials of my fellow story tellers.

Simon Singh is a Ph. D. in particle physics, mathematician and a member of BBC science department for many years.

John Lynch wrote the foreword to Simon's book and co-produced the Horizon Series film based on the book.

Why would I choose this story - the story of the search for the solution to the world's hardest ever mathematics problem - to share the platform with my stories from everyday classrooms? Simon and John can answer that:

Slides 6, 7

John Lynch, Foreword, p. ix

For some time in my research I looked for a reason why the Last Theorem mattered to anyone but a mathematician, and why it would be important to make a program about it. Mathematics has a multitude of practical applications, but in the case of number theory the most exciting uses that I

> was offered were in cryptography, in the design of acoustic baffling, and in communication from distant space craft. None of these seemed likely to draw in an audience. What was far more compelling were the mathematicians themselves, and the sense of passion that they all expressed when talking of Fermat.

My hope for this address is simply that these storytellers can help us learn how to work like a mathematician. And right now I am starting a list of things I notice in today's stories that show us what it means to work like a mathematician. Can I suggest that you do that too.

Based on John's comment, my first entries are:

- People - or to steal from the title of another excellent book, Mathematics is a Human Endeavour
- Passion

I will show you the rest of my list at the end of the address.

Perhaps if we can finally grasp that a professional mathematician does NOT go into the office in the morning, turn on the computer and do all the exercises down the left-hand side of the screen, we will be in a better position to create an alternative environment for learning mathematics in ALL schools at ALL year levels.

The paradigm we work from determines our educational actions. In mathematics education the dominate paradigm, especially in secondary schools, has been inherited, and remains almost unaltered, from the mid-1800s at the introduction of compulsory, universal education.

Stating Fermat's Theorem

With those introductory comments, let's move on to the world's hardest problem.

Around 530 BC, Pythagoras, learning from the Egyptians and Babylonians who preceded him and working within a mathematical community that we call his school, proved a geometric theorem. Today, in Western cultures we call it Pythagoras' Theorem.

Photo Show 1- 5

I will demonstrate an example of it with this Task called Pythagoras Rods.

Of course this demonstration is not a proof. It is only a signpost to something that may be worthy of proof. A hint that in a right angle triangle, the square built on the hypotenuse may be the sum of the squares built on the other two sides. The Egyptians and Babylonians provided the signposts, but Pythagoras provided the proof that this was *always* true. In other words, that it is the defining property of right angled triangle.

Once proven, the theorem became true forever. There is a straight forward reconstruction of Pythagoras' general proof in an appendix of Singh's book.

Once proven the theorem became a foundation stone for further mathematical enquiry.

How many solutions are there to the rod demonstration?

How many ways are there to demonstrate Pythagoras' Theorem.

Then things start to get a bit more complex. As we have seen, to generate a square you only need to know one side (called the root of the square, or square root).

Slide 8

For example if the square root is 3 you build 3 rows of 3 , if 4, build 4 rows of 4 and if 5, build 5 rows of 5.

So, in general, if the right angle triangle has sides x and y and hypotenuse z, we can write a numerical equivalent of the geometry as:

x rows of x + y rows of y = z rows of z

or

$x^2 + y^2 = z^2$

So, Pythagoras' Theorem is a geometric one, with a number consequence. Not a number theorem with little or no associated image. It's about squares.

But what happens if we create pentagons, hexagons or even circles on each side of the right triangle? Will the theorem still work?

Slide 9

What happens if we create cubes on each side of the right triangle? To generate a cube we also only need one side, the cube root. For example: 3 rows of 3 stacked 3 high. Could it be true that:

$x^3 + y^3 = z^3$???

Slide 10

True or not, could there be *x*, *y*, *z* that solve:

$x^4 + y^4 = z^4$???

$x^5 + y^5 = z^5$???

...

$x^n + y^n = z^n$???

Mathematicians have played with all these extensions of Pythagoras' theorem, but this last one is where Fermat gets into the act. Through mathematical history from 530BC to 1637AD, a proposition arose that:

Slide 11

For $n > 2$, there are no *x*, *y*, *z* such that:

$x^n + y^n = z^n$

Introducing Andrew Wiles

DVD

But who is this Andrew Wiles?

(Skip text here when showing DVD)

> In 1963, when he was ten years old, Andrew Wiles was already fascinated by mathematics. "I loved doing the problems in school. I'd take them home and make up new ones of my own. But the best problem I ever found I discovered in my local library." ... Andrew was drawn to a book with only one problem, and no solution. The book was *The Last Problem* by Eric Temple Bell.
>
> p. 5

> "It looked so simple, and yet all the great mathematicians in history couldn't solve it. Here was a problem that I, a ten-year-old, could understand and I knew from that moment that I would never let it go. I had to solve it."
>
> p. 6

History of Solution

My plan now is to try to summarise the key elements of the plot of our story.

> To DISPROVE Fermat's Last Theorem, mathematicians only had to find one set of x, y, z and n that WOULD work, because Fermat claimed there were no such sets.
>
> To PROVE Fermat's Last Theorem mathematicians only had to show that for an infinity of possible sets of x, y, z and n the equation WOULD NOT work.

By Andrew's time, 2000 years of history had not yet turned up a case that would work. The second option of proving every possibility looked a bit daunting, so mathematicians applied their strategy of trying a simpler case. I will use excerpts from Singh's book to outline some of the history. Remember, my proposition is that by becoming aware of the way professional mathematicians work, we may be empowered to create a mathematics learning environment that is more relevant, challenging, satisfying and fruitful for our students.

The following are quotes:

> A century after Fermat's death there existed proofs for only two specific cases of the Last Theorem. Fermat had given mathematicians a head start by providing them with the proof that there were no solutions to the equation
>
> $x^4 + y^4 = z^4$
>
> Euler had adapted the proof to show that there were no solutions to
>
> $x^3 + y^3 = z^3$
>
> After Euler's breakthrough it was necessary to prove that there were no whole number solutions to an infinity of other equations:
>
> ...
>
> Although mathematicians were making embarrassingly slow progress, the situation was not quite as bad as it might seem at first sight.
>
> ...
>
> The proof for case $n = 3$ is particularly significant because the number 3 is an example of a *prime number*.
>
> ...

> This seems to lead to a remarkable breakthrough. To prove Fermat's last theorem for all values of n, one merely has to prove it for the prime values of *n*. All other cases are simply multiples of the prime cases and would be proved implicitly.
>
> p.88 - 90 (excerpts)

In terms of my proposition, we can already we add to our list that mathematicians:

- learn from each other
- build on each other's work
- engage in high order thinking
- are content with partial solutions
- expect to take time to solve a problem

Now the story continues through the work of a great female mathematician, who had to pretend she was male to be published. Sophie Germain who the first to succeed with a general proof as opposed to proving particular cases. Again quoting from Singh:

> By the beginning of the nineteenth century, Fermat's Last Theorem had already established itself as the most notorious problem in number theory. Since Euler's breakthrough there had been no further progress, but a dramatic announcement by [Sophie Germain] was to invigorate the pursuit of Fermat's lost proof.
>
> p.97
>
> In her letter to Gauss she outlined a calculation that focused on a particular type of prime number *p* such that $(2p + 1)$ is also prime. … For values of *n* equal to these Germain primes, she used an elegant argument to show that there were probably no solutions to the equation
>
> $x^n + y^n = z^n$
>
> In 1825 her method claimed its first complete success thanks to the work of [Dirichlet and Legendre] ... Both of them independently were able to prove that the case $n = 5$ has no solutions, but they based their proofs on, and owed their success to, Sophie Germain.

> Fourteen years later the French made another breakthrough. Gabriel Lamé made some further ingenious additions to Germain's method and proved the case for the prime $n = 7$.
>
> p.106 (excerpts)

So now the problem had been trimmed even further. Germain's method disposed of a class of prime numbers and only the irregular primes were left.

> [In 1847] Kummer pointed out that there was no known mathematics that could tackle all these irregular primes in one fell swoop. However he did believe that … they could be dealt with one by one … [But] disposing of them individually would occupy the world's community of mathematicians until the end of time.
>
> Kummer had demonstrated that a complete proof of Fermat's Last Theorem was beyond the current mathematical approaches.
>
> pp.116 & 117

Kummer's work was also to add a different layer to what was becoming the romance of Fermat's Last Theorem. However, you need to read the book for yourself to discover the subplot about the wealthy industrialist jilted by a beautiful lover, how Kummer's work prevented his honourable suicide and the establishment of a very rich prize for the first to solve the problem. Begin reading at Page 121.

From around Kummer's time until the late 1940s, mathematicians were basically stumped by Fermat. However, there was no implication that they couldn't do mathematics. They just put the problem aside for now and worked on different problems. Singh continues:

> With the arrival of the computer awkward cases of Fermat's Last Theorem (those of the irregular primes) could be dispatched with speed, and after the Second World War teams of computer scientists and mathematicians proved the Last Theorem for values of n up to 500, then 1000, and then 10,000. In the 1980s Samuel S. Wagstaff of the University of Illinois raised the limit to 25,000, and more recently mathematician's could

> claim that Fermat's Last theorem was true for all values of *n* up to 4 million.
>
> p. 158

However mathematicians knew that this success was merely cosmetic. Demonstrating something up to the umpteenth million case does not prove anything about umpteen million and one. As recently as 1988 this was point was driven home again.

Slides 12 & 13

> Euler's Conjecture and Elkies' Disproof

Well you might like to check the units digit at least. The really annoying thing after 200 years of non-success is that Elkies went on to prove there were an infinite number of solutions to Euler's Conjecture.

Now let's return to the story of our 10 year old boy, as told by Simon Singh:

> For over two centuries every attempt to rediscover the proof of Fermat's Last Theorem had ended in failure. Throughout his teenage years Andrew Wiles had studied the work of Euler, Germain, Cauchy, Lamé and finally Kummer. He hoped he could learn by their mistakes, but by the time he was an undergraduate at the University of Oxford he confronted the same brick wall that faced Kummer.
>
> p.118
>
> Wiles was not prepared to give up: Finding a proof of the Last Theorem had turned from a childhood fascination into a full-fledged obsession. Having learned all there was to learn about the mathematics of the nineteenth century, Wiles decided to arm himself with techniques of the twentieth century.
>
> p.119

As with all great stories the plot becomes more complex the deeper you get into the story. I don't intend to detail every fascinating step from here. Suffice it to say that:

- Unknown to Andrew at the time, the tool he needed was the Taniyama-Shimura conjecture that was proposed by two Japanese university students following the second world war.

- It turned out that to prove Fermat he only had to prove the Taniyama - Shimura conjecture.
- For seven years he worked alone in his attic during the working day. He went to sleep with Fermat and woke up with Fermat ... but somewhere in between he also married and had children.
- He relearned every piece of mathematics that had ever been applied to Fermat and when he couldn't make any of it work any better than anyone else, he went hunting for new techniques.
- All the time he was doing mathematics because he was *trying to solve a problem*. That fact that he did not *get the answer* was no judgement on whether he could do mathematics.
- His perseverance paid off and on June 23rd, 1993 he presented his proof in a series of lectures to a world assembly of mathematicians at Cambridge, England.

He appeared on the cover of Time magazine and was lauded in newspapers and magazines all over the world.

But he had made a mistake.

Three months after the lecture, a colleague pointed out the error in his logic. The mathematics community allowed him time and opportunity to correct it. Not so easy!

For fourteen more months he hid himself away, this time with a colleague and worked through every step to correct the problem. He was frustrated time and again because neither of his main tools, called the Kolyvagin-Flach method and Iwasawa theory, were achieving what his intuition thought they should. Then:

Read p. 274 - p.275

Is it possible that we can create happy, healthy, cheerful, productive, inspiring classrooms where all students can experience that same joy of discovery?

Working Mathematically Curriculum

The answer to that question is an unequivocal YES.

Photo Show 6

If time introduce *Square Pairs*, Task 216 or Maths300 Lesson 140, and set the challenge of which party sizes will square pair. Point out that this problem, which is very much suited to school children, was suggested as so by two modern mathematicians, Johnston Anderson & Andy Walker in their 1999 paper on the problem in the Mathematical Gazette.

Coming Together

So how does solving the world's hardest problem inform the teaching of mathematics? Check the list you have been making through this address. But before I show you my list, just a few final quotes from our book:

> Mathematics is not a careful march down a well-cleared highway, but a journey into a strange wilderness, where the explorers often get lost.
>
> p. 71 *W. S. Anglin*
>
> During the era of Fermat, mathematicians were considered amateur number-jugglers, but by the eighteenth century they were treated as professional problem-solvers.
>
> p. 73
>
> Pure mathematicians just love a challenge. They love unsolved problems. When doing math there's this great feeling. You start with a problem that just mystifies you. You can't understand it, it's so complicated, you just can't make head nor tail of it. But then when you finally resolve it, you have this incredible feeling of how beautiful it is, how it all fits together so elegantly.
>
> p. 146 *Andrew Wiles*
>
> Mathematics has its applications in science and technology, but that is not what drives mathematicians. They are inspired by the joy of discovery.
>
> p. 146

Do our classrooms offer this joyful opportunity to all students, or are they more likely to reflect Ancient Egyptian thinking:

> Pythagoras observed that the Egyptians and Babylonians conducted each calculation in the form of a recipe that could be followed blindly. The recipes, which would have been passed down through generations, always gave the correct answer and so nobody bothered to question them or explore the logic underlying the equations. What was important for these civilizations was that a calculation worked - why it worked was irrelevant.
>
> p. 7

Slides 14 - 18

And now to my list.

And if this list doesn't inform your mathematics teaching, then I am not sure what will.

Slide 19

But how to remember it? Usually we remember better if we understand the principles from which the elements derive. So, as the finale to this performance (sorry keynote address), let me help. Feel free to join in at any time to save me from myself.

Working Mathematically: The Anthem
(Tune: Advance Australia Fair)

Maths teachers all let us rejoice

Our subject is not trite!

It's far more than the daily toil

Of "Is this wrong or right?"

The theme we weave each time we teach

Must challenge students to

Engage with problems in the way

That math'maticians do!

In joyful classrooms let us work

Like math'maticians do!

References

Singh, Simon (1997) *Fermat's Last Theorem* (UK), *Fermat's Enigma* (USA), Anchor Books (UK), Anchor Books/Walker Books (USA)

DVD, *Fermat's Last Theorem*, BBC Horizon. Available through Utube in three parts, but for a real copy try BBC Active, Sydney, NSW, 02 9454 2390. In the USA the DVD is available through PBS in the Nova series as *The Proof*.